APPLICATION OF AN EQUIVALENT PROBLEM IN HEAT CONDUCTION THEORY TO THE CALCULATION OF SEMIUNBOUNDED TURBULENT JETS

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We indicate the possibility of using the well-known method of the equivalent problem in heat conduction theory to calculate turbulent wall jets. The method is used to calculate the velocity profiles and the excess heat content at cross sections and also the friction drag and heat transfer coefficients for semiunbounded jets.

Among the various computational schemes based on replacing the boundary layer equations by a linear heat conduction equation, the method of the so-called equivalent problem in heat conduction theory [1] has been widely used.

The method has been successfully applied to calculate many practically important cases in complex jet flows and gas jets. In particular, problems in the propagation of isothermal jets of finite dimension, the diffusion of a gas jet in a stationary medium and in a slipstream, a three-dimensional jet, the development of a jet with complex initial velocity profile, etc., have been solved.

We consider the possibility of applying this method to the investigation of semiunbounded (wall) turbulent jets.

Following [1], we transform from the actual coordinates x and y to certain (derived) coordinates ξ and η such that the field of the jet flow is described by an equation of the form

$$\frac{\partial L}{\partial \xi} = \frac{\partial^2 L}{\partial \eta^2}, \tag{1}$$

where $L = \rho u^2$ for a dynamic, and $L = \rho u c_p \Delta t$ for a thermal problem.

We know that for the class of free flows of liquid and gas jets the condition for the change of the variables leading to an equation of the heat conduction type has the form $\xi_i = \xi(x)$ (i = 1 and i = 2 respectively for a dynamic and a thermal problem), $\eta \approx y$.

If we use the equivalent problem in heat conduction theory to calculate semiunbounded turbulent jets, as distinct from free jets, we have to deform both the x and y coordinates. Then, as experience shows, $\xi_i = \xi(x)$, $\eta_i = \eta(y)$ and the connection between the derived coordinates η_i and y can be established starting from the equation $\eta_i = y^{\alpha/2}$ (strictly for the self-similar part of the jet), for $\xi_1 = cx^{\alpha}$.

Equation (1) is solved under the same boundary conditions for which the boundary layer equation has to be integrated. For example, for a dynamic problem, the initial and boundary conditions have the form

$$\xi_1 = 0\ (x = 0)\ \begin{cases} L = \rho_0 u_0^2 & \text{for}\ \ 0 \leqslant \eta_1 \leqslant b_0, \\ L = 0 & \text{for}\ \ b_0 \leqslant \eta_1 \leqslant \infty, \end{cases} \tag{2}$$
$$\xi_1 > 0\ \ (x > 0)\ \ L = 0\ \ \text{for}\ \ \eta = 0\ \text{and}\ \eta_1 = \infty$$

for a semiunbounded submerged jet and

$$\xi_1 = 0\ \begin{cases} L = \rho_0 u_0^2 & \text{for}\ \ 0 \leqslant \eta_1 \leqslant b_0, \\ L = \rho_\infty u_\infty^2 & \text{for}\ \ b_0 \leqslant \eta_1 \leqslant \infty, \end{cases}$$

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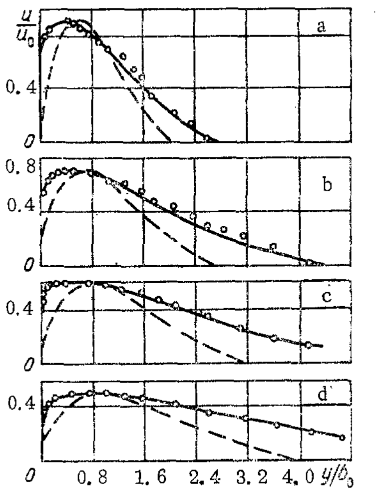

Fig. 1

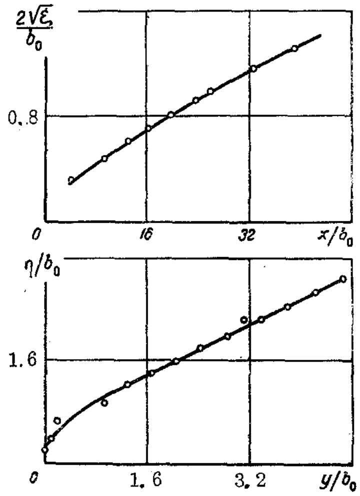

Fig. 2

Fig. 1. Comparison of the computed and experimental velocity profiles in a semiunbounded jet (m = 0): a) $x/b_0 = 9.7$; b) $x/b_0 = 19.5$; c) $x/b_0 = 26.9$; d) $x/b_0 = 39.0$.

Fig. 2. The derived coordinates ξ and η as functions of x and y.

$$\xi_1 > 0 \begin{cases} L = 0 & \text{for} \quad \eta_1 = 0, \\ L = \rho_\infty u_\infty^2 & \text{for} \quad \eta_1 = \infty \end{cases} \tag{3}$$

for a semiunbounded jet developing in a slipstream.

The solution of Eq. (1) for the initial and boundary conditions (2) and (3) is known [2]:

$$\frac{\rho u^2}{\rho_0 u_0^2} = \frac{m-1}{2}\left[\operatorname{erf}\left(\frac{\eta_1 - b_0}{2\sqrt{\xi_1}}\right) + \operatorname{erf}\left(\frac{\eta_1 + b_0}{2\sqrt{\xi_1}}\right)\right] + \operatorname{erf}\left(\frac{\eta_1}{2\sqrt{\xi_1}}\right), \tag{4}$$

where $\operatorname{erf}(t) = (2/\sqrt{\pi})\int_0^t e^{-z^2}dz$ is the error integral; $m = \rho_\infty u_\infty^2/\rho_0 u_0^2$ is the cocurrent parameter.

We note that (4) was obtained under the assumption that the momentum flux distribution in the initial cross section of the jet is uniform. However, the presence of a boundary layer at the surface and at the pipe walls in the actual jet outflow prevents us from obtaining a uniform profile of ρu^2. Hence, strictly speaking, the momentum flux distribution density at various distances from the end of the pipe has to be calculated taking account of the initial nonuniformity in the profile of $\rho_0 u_0^2$.

We transform from $\rho u^2/\rho_0 u_0^2$ directly to velocities and temperatures by the usual algebraic conversion [1]. The transformation equation $\xi = f(x)$ is defined from experiment by comparing the computed and experimental curves for the variation in the maximum value of ρu^2 along the length of a flat plate.

Experiments on the study of the law of propagation of a wall jet in a hot slipstream were made for $0 \leq m \leq 1$ and $1 \leq w \leq 3.0$, where w is the nonisothermicity parameter. A detailed description of the apparatus, method of measurement, and analysis of the experimental results, and also the fundamental results of the experiments are given in [3, 5].

Let us compare the results of the computation for m = 0, using (4), with the experimental results of [5] to indicate the possibility of applying the method of the equivalent problem in heat conduction theory to the calculation of semiunbounded turbulent jets.

Figure 1 shows the distribution of u/u_0, obtained experimentally, at various distances from the end of the pipe and compares them with the values computed from (4). The dotted lines show the computed velocity profile $u/u_0 = f(\eta_1)$ when only one derived coordinate ξ is deformed, and the continuous lines show the same velocity profile when we take account of the equation $\eta_1 = \eta_1(y)$.

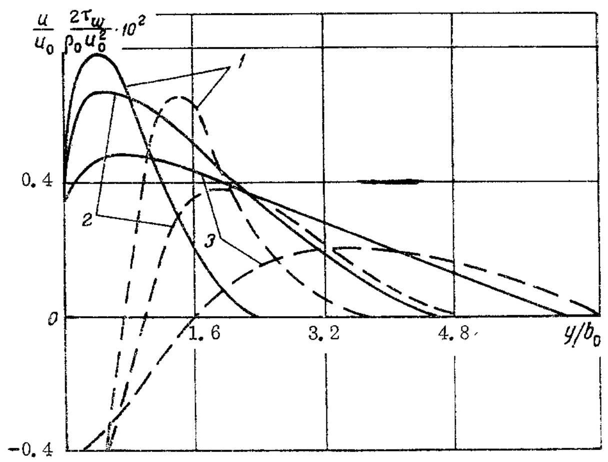

Fig. 3. Tangential friction stress distribution at sections of a semiunbounded jet (m = 0): 1) $x/b_0 = 9.72$; 2) $x/b_0 = 19.5$; 3) $x/b_0 = 39$.

We see from the graphs that when we take note of the equation $\eta_1 = \eta_1(y)$, computations based on (4) agree satisfactorily with the experimental data (points).

Figure 2 shows the equations $\sqrt{\xi_1}/b_0 = f(x/b_0)$ and $\eta_1/b_0 = f_1(y/b_0)$ obtained by comparing the solution of Eq. (1) with experiment. The values of α and β obtained, assuming that $\sqrt{\xi_1}/b_0 = c(x/b_0)^\alpha$ and $\eta_1/b_0 = (y/b_0)^\beta$, from the above equations were equal to 2/3. This experimental fact implies that the method of the equivalent problem in heat conduction theory can be used to calculate semiunbounded jets essentially when only one (longitudinal) coordinate is deformed. Then it is sufficient to determine the second coordinate in the first approximation from the equation $\eta_1 = y^{\alpha/2}$ if $\xi_1 = cx^\alpha$.

In turbulent semiunbounded jets, in addition to the velocity profile (excess temperature, concentration of matter), the greatest interest is in the distribution of the tangential friction stresses and the flux density throughout the mixing zone, the variation in the friction drag coefficient (heat transfer) along the length of the wall, etc.

Comparing Eq. (1) with the boundary layer equation (in real variables) we can obtain the following expression for the nondimensional tangential friction stress:

$$\frac{\tau}{(\rho u^2)_0} = \overline{\rho u v} - \frac{d\bar{\xi}_1}{d\bar{x}} \int_{\eta}^{\infty} \frac{\partial^2}{\partial \bar{\eta}_1^2} (\overline{\rho u^2}) \frac{d\bar{y}}{d\bar{\eta}_1}\, d\bar{\eta}, \tag{5}$$

where $\overline{\rho uv} = \rho uv/(\rho u^2)_0$, $\bar{x} = x/b_0$, $\bar{\xi}_1 = \xi_1/b_0$, $\bar{y} = y/b_0$, $\bar{\eta}_1 = \eta_1/b_0$ are nondimensional variables; v is the transverse velocity component defined by the equation of continuity.

We note that when $\xi_1 = \xi(x)$ and $\eta \approx y$, Eq. (5) easily transforms into the familiar equation for turbulent friction obtained in [1].

From (5) we calculated the distributions of the tangential friction stresses at a cross section of a submerged (m = 0) semiunbounded jet for three values of x/b_0. In doing so we used (4) and the equations $\xi_1(x)$ and $\eta_1(y)$ (Fig. 2) found from experiment. The results of the calculation are shown in Fig. 3. The continuous lines show the nondimensional velocity profiles and the dotted lines the tangential friction stress distributions.

We see from the graph that the law for the change in the tangential friction stresses at cross sections of a semiunbounded jet close to the wall is nearly linear, while far from the wall it is similar to the law for the change in $\tau/(\rho u^2)_0$ for a free submerged jet. It is noteworthy that the turbulent friction $\tau = -\rho u'v'$, computed by the method of the equivalent problem in heat conduction theory (Eq. (5)) is nonzero at the point where the velocity is at its maximum. We note that the fact that the turbulent friction is nonzero at the point of maximum velocity in semiunbounded jets has been remarked in other papers [4, 6, etc.].

For $\eta = 0$ ($y = 0$), Eq. (5) yields the value of the tangential friction stress at the wall:

$$C_{f/2} = \frac{\tau_w}{(\rho u^2)_0} = -\frac{d\xi_1}{dx}\int_0^\infty \frac{\partial^2}{\partial y^2}(\rho u^2)\frac{dy}{d\eta_1}\,d\eta_1. \tag{6}$$

The results of calculating the drag coefficient from this equation are given in Fig. 4. Figure 4 also gives a comparison of the experimental results of several authors [7, 8, etc.]. The continuous lines in Fig. 4 show the results of direct measurements of the drag coefficient by Preston's method (curves 2, 4) and of a heated film (curve 3), while the dotted lines show the calculation of C_f from the Reynolds analogy and the heat transfer data (curves 5, 6) and by the method of integral ratios (curve 1).

We see from the graph that the results of the calculation (curve 7) agree satisfactorily with the data of other authors.

Comparing (1) for a heat flux ($L = \rho u c_p \Delta T$) with the energy equation for a moving liquid, as was done for the friction stress and the friction drag coefficient, we can obtain computational equations for the heat flux distribution at a cross section of a wall jet and for the heat transfer along the length of a flat plate. Thus, for example, the specific heat flux at the wall ($\eta_2 = y = 0$) is defined by the equation

$$\frac{q_w}{\rho_0 u_0 c_p \Delta T} = -\frac{d\xi_2}{dx}\int_0^\infty \frac{\partial^2}{\partial \eta_2^2}(\rho u c_p \Delta T)\frac{dy}{d\eta_2}\,d\eta_2, \tag{7}$$

where $\xi(x)$ and $\eta_2(y)$ are the derived coordinates, to be defined by experiment, differing in general from ξ_1 and η_1. However, as the results of analyzing the experimental data on the distribution of the excess heat content at a cross section of the jet showed, in the first approximation we can take $\xi_2 \approx \xi_1$ and $\eta_2 = \eta_1$. Taking account of this, from the equation derived above for the Stanton number, we obtain the following equation:

$$\mathrm{St} = \left(\frac{T_0 - T_\infty}{T_0 - T_w}\right)\left[-\frac{d\xi_1}{dx}\int_0^\infty \frac{\partial^2}{\partial \eta_1^2}(\rho u c_p \Delta T)\frac{dy}{d\eta_1}\,d\eta_1\right]. \tag{8}$$

The results of calculating the nondimensional heat transfer coefficient from this equation are shown in Fig. 5. Figure 5 also shows the experimental equation $\mathrm{Nu} = f(\mathrm{Re}, \bar{x}, T)$ from various papers and also the results of analyzing the experimental results (of this and other investigations [4, 7-9]).

We see from the graph that the values of the heat transfer coefficient (curve 3), computed from the method of the equivalent problem in heat conduction theory, under the assumption that $\xi_2(x) = \xi_1(x)$, deviate

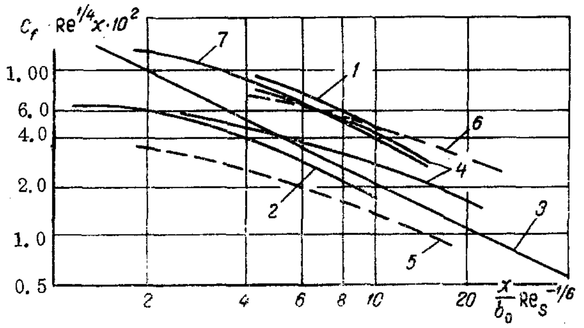

Fig. 4

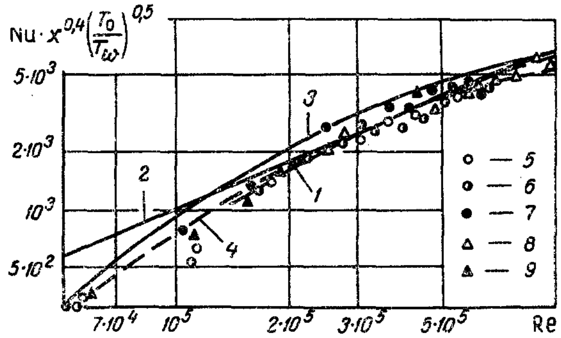

Fig. 5

Fig. 4. Variation in the friction drag coefficient along a flat plate in a wall jet ($m = 0$): 1) from the data of [11]; 2) from the data of [7]; 3) from the data of [8]; 4) from the data of [12]; 5) from the data of [10]; 6) from the data of [9]; 7) computation by the method of the equivalent problem in heat conduction theory (Eq. (6)).

Fig. 5. Variation of the heat transfer coefficient along the length of a flat plate in a nonisothermal jet ($m = 0$): 1) from the data of [7]; 2) from the data of [9]; 3 and 4) calculation by the method of the equivalent problem in heat conduction theory with $\mathrm{Pr_T} = 1$ and 0.8 respectively; 5-7 and 8, 9) experimental data from [5] and [4]; 5) $T_0/T_w = 1.10$; 6) $T_0/T_w = 1.64$; 7) $T_0/T_w = 2.44$; 8) $T_0/T_w = 0.88$; 9) $T_0/T_w = 1.08$.

systematically from the experimental results. A better correspondence between the results is obtained if we assume, as for free jet flows [1] that $\xi_1(x)/\xi_2(x) = Pr_T \approx 0.8$ (curve 4).

Thus, the comparisons in Figs. 1-5 of the results of calculating the velocity profiles and the excess heat content at various cross sections, and also the friction drag and heat transfer coefficients, with the experimental results show that the method of the equivalent problem in heat conduction theory can successfully be used to calculate the local and integral characteristics of a semiunbounded jet.

The method we have discussed is promising for the calculation of wall jets propagating in a nonisothermal slipstream, for calculating the effectiveness of a gas screen etc. It makes it possible to generalize the experimental material and carry out a more detailed analysis of the experimental results.

NOTATION

ρ	is the density, kg/m^3;
u	is the velocity, m/sec;
c_p	is the heat capacity, J/deg;
T_0, T_∞, T_W, T	is the jet temperature at the exit from the pipe, in the surrounding medium, at the wall and the current temperature, °K;
x, y	are the longitudinal and transverse coordinates of the boundary layer, m;
η, ξ	are the derived coordinates, defined from experiment;
b_0	is the width of exit cross section, m;
τ	is the friction stress, N/m^2;
q_W	is the heat flux at wall, J/m^2;
C_f	is the friction drag coefficient;
St	is the Stanton number;
$Nu = \alpha x/\lambda$	is the Nusselt number;
$Re = u_0 x/\nu_0$	is the Reynolds number;
$\bar{x} = x/b_0$	is the nondimensional distance from the end of the pipe;
m	is the current parameter.

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